Chapter 12

Vector Valued Functions and Motions in Space

12.1 Curves and Tangents

When a particles moves in the space during a times interval $I = [a, b]$, we think of its **coordinates** as a vector function

$$\mathbf{r}(t) = (f(t), g(t), h(t))$$

defined on I. The points $(x, y, z) = (f(t), g(t), h(t))$ make up a curve called a **path**.

Definition 12.1.1. A **curve**(or **path**) can be represented as a function $\mathbf{r} : I \to \mathbb{R}^n, n = 2, 3$. It is called a **parameterized curve**. $\mathbf{r}(a)$ and $\mathbf{r}(b)$ are called the **endpoints** of the path.

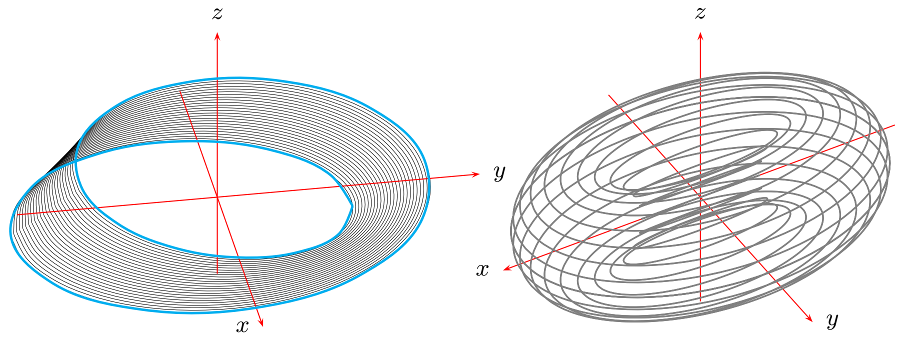
A parameterized curve $\mathbf{r}$ in $\mathbb{R}^3$ (or $\mathbb{R}^2$) can be written as

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} = (x(t), y(t), z(t)). \tag{12.1}$$

Here, $f(t), g(t), h(t)$ are called **component functions**. It may be viewed as the position of a particle moving along the curve.

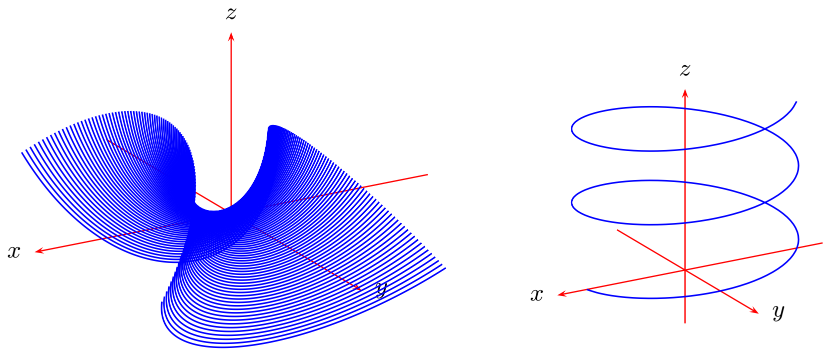
A function having vector values, like equation (12.1) is called a **vector valued function**.

Example 12.1.2. (1) $\mathbf{r}(t) = \mathbf{a} + t\mathbf{b}$ is a line



$\left(\left(1+\frac{1}{2}\cos\frac{t}{2}\right)\cos t,\ \left(1+\frac{1}{2}\cos\frac{t}{2}\right)\sin t,\ \frac{1}{2}\sin\left(\frac{t}{2}\right)\right)$ $((a+b\cos 20t)\cos u, (a+b\cos 20t)\sin u, b\sin 20t)$

Figure 12.1: Graph of Möbius strip and torus



Enneper's Surface $(u-\frac{u^3}{3}+uv^2, v-\frac{v^3}{3}+vu^2, u^2-v^2)$

Helix : $(\cos t, \sin t, t)$.

Figure 12.2: Family of curves

(2) $\mathbf{r}(t) = (\cos t, \sin t)$ on $[0, 2\pi]$ is path traveling a circle once. If the domain is $[0, 4\pi]$, it travels twice.

(3) A family of curves are obtained from surface: If we fix say $v = 1$ from Enneper's surface, we get $(2u - \frac{u^3}{3}, \frac{2}{3} + u^2, u^2 - 1)$. (Fig 12.2)

(4) $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$ defines a circular helix. (Fig 12.2)

A path may have many **parametrizations**.

Limits and continuity

Definition 12.1.3. We define the limit of a vector function as

$$\lim_{t\to t_0} \mathbf{r}(t) = \mathbf{L} = (\lim_{t\to t_0} f(t), \lim_{t\to t_0} g(t), \lim_{t\to t_0} h(t)).$$

One may use ϵ-δ to define the limit of each component.

A vector function $\mathbf{r}(t)$ is **continuous at a point** t_0 if $\lim_{t\to t_0}\mathbf{r}(t) = \mathbf{r}(t_0)$. The function $\mathbf{r}(t)$ is **continuous** if the function is continuous at all points of its definition.

Derivatives and Motion

Definition 12.1.4. If all the components of $\mathbf{r}(t)$ is differentiable, then we say $\mathbf{r}(t)$ is **differentiable** and its derivative is written as

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} == \lim_{\Delta t\to 0}\frac{\mathbf{r}(t+\Delta t)-\mathbf{r}(t)}{\Delta t} = (f'(t), g'(t), h'(t)). \qquad (12.2)$$

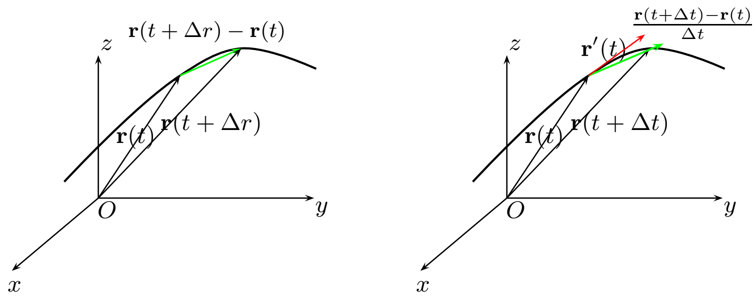


Figure 12.3: As $\Delta t \to 0$, $\mathbf{r}'(t)$ becomes tangent vector

The geometric meaning of derivative of $\mathbf{r}(t)$

When $\mathbf{r}'(t) \neq 0$, it represents a **tangent vector** at t.

Definition 12.1.5. A curve $\mathbf{r}(t)$ is said to be **smooth** if $d\mathbf{r}/dt$ is continuous and never zero. In this case, the image curve looks smooth. One of the reason for requiring nonzero derivative is that we want to avoid the case when a particle moving along the curve traces back. (i.e., move backward)

On a smooth curve, there is no sharp corner or cusps.

Definition 12.1.6. Let $\mathbf{r}$ be a smooth curve. Then

(1) the **velocity** is defined : $\mathbf{v}(t) = \mathbf{r}'(t)$

(2) the **speed** of $\mathbf{r}$ is $\|\mathbf{v}(t)\|$.

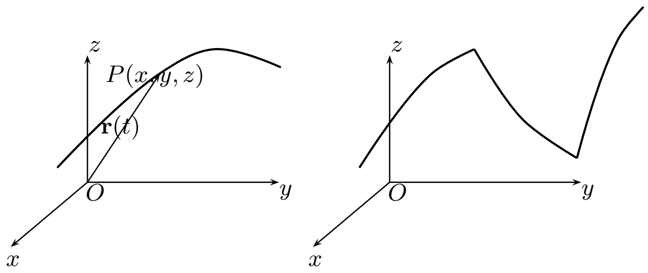


Figure 12.4: piecewise smooth curve can have no tangent at cusps

(3) the **acceleration** vector is $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

(4) the unit vector $\mathbf{v}(t)/\|\mathbf{v}(t)\|$ is the direction.

Example 12.1.7. Find the velocity, acceleration of a particle moving along the curve $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 4\cos^2 t\mathbf{k}$.

sol.

$$\mathbf{v}(t) = \mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} - 8\cos t\sin t\mathbf{k} = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} - 4\sin 2t\mathbf{k}.$$

The acceleration is

$$\mathbf{a}(t) = \mathbf{r}''(t) = -2\cos t\mathbf{i} - 2\sin t\mathbf{j} - 8\cos 2t\mathbf{k}$$

The speed is

$$\|\mathbf{v}(t)\| = \sqrt{(-2\sin t)^2 + (2\cos t)^2 + (-4\sin 2t)^2} = \sqrt{4 + 16\sin^2 t}.$$

The position when $t = 7\pi/4$ is

$$\mathbf{r}(\frac{7\pi}{4}) = 2\cos\frac{7\pi}{4}\mathbf{i} + 2\sin\frac{7\pi}{4}\mathbf{j} + 4\cos^2\frac{7\pi}{4}\mathbf{k} = \sqrt{2}\mathbf{i} - \sqrt{2}\mathbf{j} + 2\mathbf{k}.$$

The velocity vector at $t = 7\pi/4$ is

$$\mathbf{v}(\frac{7\pi}{4}) = \sqrt{2}\mathbf{i} + \sqrt{2}\mathbf{j} + 4\mathbf{k}.$$

□

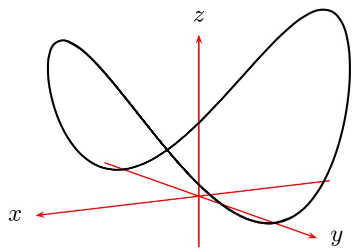


Figure 12.5: Curve of Example 12.1.7

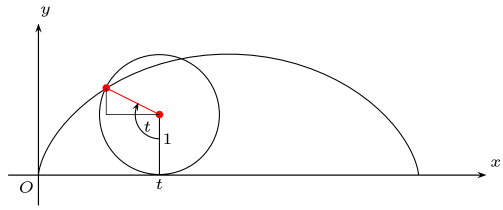


Figure 12.6: cycloid

Example 12.1.8. A particle moves with a constant acceleration $\mathbf{a}(t) = -\mathbf{k}$. When $t = 0$ is the position is $(0, 0, 1)$ and velocity is $\mathbf{i}+\mathbf{j}$. Describe the motion of the particle.

sol. Let $\mathbf{c}(t) = (x(t), y(t), z(t))$ represent the path traveled by the particle. Since the acceleration is $\mathbf{c}''(t) = -\mathbf{k}$ we see the velocity is

$$\mathbf{c}'(t) = C_1\mathbf{i} + C_2\mathbf{j} - t\mathbf{k} + C_3\mathbf{k}.$$

Hence by initial condition, $\mathbf{c}'(t) = \mathbf{i} + \mathbf{j} - t\mathbf{k}$ and so $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} - \frac{t^2}{2}\mathbf{k} + Const\ vec$. The constant vector is $\mathbf{k}$. Hence $\mathbf{c}(t) = t\mathbf{i} + t\mathbf{j} + (1 - \frac{t^2}{2})\mathbf{k}$.

□

Example 12.1.9. The image of C^1-curve is not necessarily "smooth". it may have sharp edges; (Fig 12.7)

(1) Cycloid: $\mathbf{c}(t) = (t - \sin t, 1 - \cos t)$ has cusps when it touches x-axis. That is, when $\cos t = 1$ or when $t = 2\pi n, n = 1, 2, 3, \cdots$.

(2) Hypocycloid: (Fig 12.8) $\mathbf{c}(t) = (\cos^3 t, \sin^3 t)$ has cusps at four points when $\cos t = 0, \pm 1$.

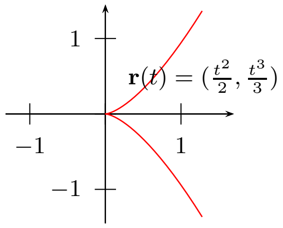


Figure 12.7: At a cusp, $\frac{d\mathbf{r}(t)}{dt}|_{t=0} = 0$

(3) Consider $\mathbf{r}(t) = (\frac{t^2}{2}, \frac{t^3}{3})$. Eliminating t, we get

$$(2x)^3 = (3y)^2.$$

At all these points, we can check that $\mathbf{c}'(t) = 0$.(Roughly speaking, tangent vector has no direction or does not exist.)

Proposition 12.1.10. *Let* $\mathbf{r}$ *be a differentiable path and assume* $\mathbf{v}_0 = \mathbf{v}(t_0) \neq 0$. *The tangent line to the path is given by*

$$\ell(t) = \mathbf{r}_0 + (t - t_0)\mathbf{v}_0. \tag{12.3}$$

Differentiation Rules

(1) $\frac{d}{dt}[\mathbf{C}] = \mathbf{0}$

(2) $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ for any diff'able scalar function $f(t)$

(3) $\frac{d}{dt}[\mathbf{u}(t) \pm \mathbf{v}(t)] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$ (Sum/difference)

(4) $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ (dot product)

(5) $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ (cross product)

(6) $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (chain rule)

Proof. Proof of cross product rule.

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times\mathbf{v}(t+h)-\mathbf{u}(t)\times\mathbf{v}(t)}{h}\\ &= \lim_{h\to 0}\frac{(\mathbf{u}(t+h)-\mathbf{u}(t))\times\mathbf{v}(t+h)+\mathbf{u}(t)\times(\mathbf{v}(t+h)-\mathbf{v}(t))}{h}\\ &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)-\mathbf{u}(t)}{h}\times\mathbf{v}(t)+\mathbf{u}(t)\lim_{h\to 0}\times\frac{\mathbf{v}(t+h)-\mathbf{v}(t)}{h}.\end{aligned}$$

Try the proof of chain rule.(Hint: do componentwise). □

Example 12.1.11. Show that if $\mathbf{r}(t)$ is a vector function such that $\|\mathbf{r}(t)\|$ is constant, then $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$ for all t.

Solution: [Draw a curve on the unit sphere.]
$\|\mathbf{r}(t)\|^2 = \mathbf{r}(t)\cdot\mathbf{r}(t)$. Derivative of constant is zero. Hence

$$0 = \frac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \mathbf{r}'(t)\cdot\mathbf{r}(t)+\mathbf{r}(t)\cdot\mathbf{r}'(t) = 2\mathbf{r}(t)\cdot\mathbf{r}'(t).$$

Thus $\mathbf{r}'(t)$ is perpendicular to $\mathbf{r}(t)$.

□

Example 12.1.12. A circle C of radius 1/4 is rolling along the unit circle $U\colon x^2+y^2=1$. Represent the locus of the point P starting from $(1,0)$ to return to itself. (Refer to Figure 12.8).

sol. The center of C is at $\frac{3}{4}(\cos t,\sin t)$, The desired point $\mathbf{c}(t)$ is given by

$$\begin{aligned}\mathbf{c}(t) &= \frac{3}{4}(\cos t,\sin t)+\frac{1}{4}(\cos(t-4t),\sin(t-4t))\\ &= \frac{1}{4}(3\cos t+\cos 3t, 3\sin t-\sin 3t)\end{aligned}$$

Since $\mathbf{c}(2\pi)=(1,0)$, we see the path is

$$\mathbf{c}(t) = \frac{1}{4}(3\cos t+\cos 3t, 3\sin t-\sin 3t),\quad 0\le t\le 2\pi.$$

By the trig. identity [1] it beceoms

$$\mathbf{c}(t) = (\cos^3 t, \sin^3 t),\quad 0\le t\le 2\pi$$

[1] $\sin 3\theta = 3\sin\theta - 4\sin^3\theta,\quad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$

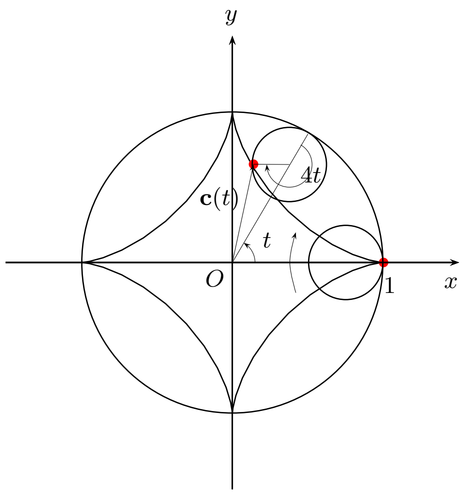


Figure 12.8: Hypocycloid $x^{2/3} + y^{2/3} = 1$

or

$$x^{2/3} + y^{2/3} = 1$$

□

12.2 Integrals of Vector functions; Projectile Motion

If the component of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ is integrable over $[a, b]$ then we can define its integral as follows

$$\boxed{\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}}$$

12.2.1 Projectile Motion

Example 12.2.1 (Throwing a ball)**.** Assume a baseball a player throws a ball(or a cannon ball) with an initial velocity $\mathbf{v}_0$m/sec that is in the direction of $(\cos\alpha, \sin\alpha)$. Describe the trajectory.

sol. The motion follows from Newton's second law of motion:

The force acting on the ball is equal to the mass times the acceleration: $\mathbf{F} = m\mathbf{a}$.

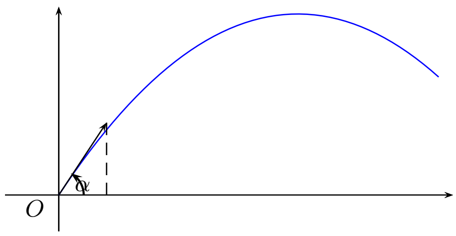


Figure 12.9: A projectile

Since the acceleration is $\mathbf{a}(t) = \mathbf{r}''(t)$, we must have

$$m\mathbf{a} = m\mathbf{r}''(t) = -mg\mathbf{j} \text{ or } \mathbf{r}''(t) = -g\mathbf{j},$$

where $g = 9.8m^2/sec$ is the gravity constant.

Integrating, we get the velocity

$$\mathbf{v}(t) := \mathbf{r}'(t) = -gt\mathbf{j} + \mathbf{c}$$

for some constant vector $\mathbf{c}$. Integrating once more, we obtain

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{c}t + \mathbf{d}.$$

Since the initial velocity is $\mathbf{v}(0) = \mathbf{c} = 20(\cos\alpha, \sin\alpha)$, $v_0 = 20$ we have

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + (v_0\cos\alpha)t\mathbf{i} + (v_0\sin\alpha)t\mathbf{j} + \mathbf{d},$$

where $\mathbf{d}$ is the initial position of the ball.

□

Projectile motion with wind gusts

Example 12.2.2 (baseball)**.** A baseball is hit when it is 1 m above the ground. The initial speed if 50m/s at an angle of 20 degrees (with horizontal). At the moment of hit, the wind was blowing in the opposite direction of the ball $2.5m\mathbf{i}/s$.

(1) Find the location of the ball

(2) How high does the ball go and when it reaches its maximum height?

(3) How far it would go until it hits the ground and when ?

sol. The situation is the same as above example except the effect of wind. So

$$\begin{aligned}\mathbf{r}(t) &= -\frac{1}{2}gt^2\mathbf{j} + (v_0\cos\alpha - 2.5)t\mathbf{i} + (v_0\sin\alpha)t\mathbf{j} + \mathbf{j}\\ &= (50\cos 20^o - 2.5)t\mathbf{i} + (1 + 50\sin 20^o t - 4.9t^2)\mathbf{j}.\end{aligned}$$

it reaches maximum when $dy/dt = 50\sin 20^o - 9.8t = 0$, $t = 1.75$.

□

12.3 Arc Length

Definition 12.3.1 (Arc Length)**.** Suppose a curve C has one-to-one differentiable parametrization $\mathbf{r}$. Then the **arc length** is defined by

$$L(\mathbf{r}) = \int_a^b \|\mathbf{v}(t)\|\,dt = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt.$$

To find the length of a path, we divide the path into small pieces and approximate each piece by a line segment joining the end points; then summing the length of individual line segments we obtain an approximate length. The length is obtained by taking the limit. To define it precisely, we use the Riemann integral.

The sum of the line segment is

$$\begin{aligned}\sum_{i=1}^k \Delta s_i &= \sum_{i=1}^k \|\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})\|\\ &= \sum_{i=1}^k \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}\\ &= \sum_{i=1}^k \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2}\Delta t_i.\end{aligned}$$

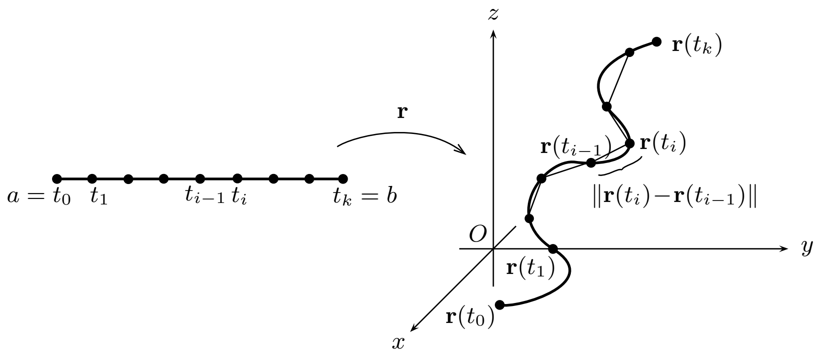


Figure 12.10: Riemann sum of the curve length

As $k \to \infty$ it converges to

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt. \tag{12.4}$$

Example 12.3.2. Find the length of the curve $x^{2/3} + y^{2/3} = 1$.

sol. It suffices to consider the first quadrant and we parameterize it as

$$\mathbf{r}(t) = (\cos^3 t, \sin^3 t), \quad 0 \le t \le \pi/2.$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} = 3|\sin t \cos t|$$

Length is

$$\begin{aligned} 4\int_0^{\pi/2} 3|\sin t \cos t|\, dt &= 6\int_0^{\pi/2} \sin 2t\, dt \\ &= 6\left[-\frac{1}{2}\cos 2t\right]_0^{\pi/2} \\ &= 6\left[-\frac{1}{2}(-1) - \left(-\frac{1}{2}\right)\right] \\ &= 6 \end{aligned}$$

□

Example 12.3.3. Find the arclength of the helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, $0 \le t \le 2\pi$.

Sol.

$$\|\mathbf{r}'(t)\| = \| - a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}\| = \sqrt{a^2+b^2}.$$

Hence

$$L(\mathbf{r}) = \int_0^{2\pi} \sqrt{a^2+b^2}dt = 2\pi\sqrt{a^2+b^2}.$$

Example 12.3.4. Find the arclength of the curve $(\cos t, \sin t, t^2)$, $0 \le t \le 2\pi$.

Sol.

$$\|\mathbf{v}\| = \sqrt{1+4t^2} = 2\sqrt{t^2+\frac{1}{4}}.$$

To evaluate this integral we need a table of integrals:

$$\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}[x\sqrt{x^2+a^2} + a^2\log(x+\sqrt{x^2+a^2})] + C.$$

Example 12.3.5. Find the length of the cycloid

$$\mathbf{r}(t) = (t - \sin t, 1 - \cos t).$$

Since

$$\|\mathbf{r}'(t)\| = \sqrt{(1-\cos t)^2 + (\sin t)^2} = \sqrt{2 - 2\cos t}$$

we see

$$\begin{aligned} L(\mathbf{r}) &= \int_0^{2\pi} \sqrt{2-2\cos t}dt = 2\int_0^{2\pi} \sqrt{\frac{1-\cos t}{2}}\,dt \\ &= 2\int_0^{2\pi} \sin\frac{t}{2}\,dt \\ &= 4\left(-\cos\frac{t}{2}\right)\Bigg|_0^{2\pi} = 8. \end{aligned}$$

Example 12.3.6. Suppose a function $y = f(x)$ given. Then the graph is viewed as a curve parameterized by $t = x$ and $\mathbf{r}(x) = (x, f(x))$. So the length of the graph from a to b is

$$L(\mathbf{r}) = \int_a^b \sqrt{1+(f'(x))^2}dx.$$

Velocity and speed

Assume the path $\mathbf{r}(t) = (x(t), y(t), z(t))$ represents the movement of an object. In other word, the location of the object at time t is given by $\mathbf{r}(t)$. Then the

instantaneous velocity at $t = t_0$ is given as follows, and it is the tangent vector at $t = t_0$.

$$\mathbf{r}'(t_0) = \lim_{h \to 0} \frac{\mathbf{r}(t_0 + h) - \mathbf{r}(t_0)}{h} = (x'(t_0), y'(t_0), z'(t_0)).$$

Example 12.3.7. If an object follow moving along the curve $\mathbf{c}(t) = t\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k}$ at time t takes off the curve at $t = 2$ and travels for 5 seconds. Find the location.

sol. We assume the object travels along the tangent line after taking off the curve. The velocity at $t = 2$ is $\mathbf{c}'(2) = \mathbf{i} + 4\mathbf{j} + e^2\mathbf{k}$. Hence the location 5 second after taking off the curve

$$\begin{aligned}\mathbf{c}(2) + 5\mathbf{c}'(2) &= 2\mathbf{i} + 4\mathbf{j} + e^2\mathbf{k} + 5(\mathbf{i} + 4\mathbf{j} + e^2\mathbf{k}) \\ &= 7\mathbf{i} + 24\mathbf{j} + 6e^2\mathbf{k}.\end{aligned}$$

Hence the location is $(7, 24, 6e^2)$.

□

Arc-Length Parameter

Recall : Given a C^1-parametrization of a curve C. Then we have seen that the **arc length** of C is given by

$$L(\mathbf{r}) = \int_a^b \|\mathbf{r}'(t)\| \, dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2} dt.$$

Definition 12.3.8. Now we fix a base point $P = P(t_0)$ and let the upper limit be the variable t. Then the arclength becomes a function of t, called the **arc-length function** :

$$s(t) = \int_{t_0}^t \|\mathbf{r}'(\tau)\| \, d\tau.$$

The arc-length (parameter)function satisfies

$$\frac{ds}{dt} = s'(t) = \|\mathbf{r}'(t)\| = speed.$$

Assuming $\mathbf{r}'(t) \neq 0$, we see $\frac{ds}{dt}$ is always positive. Hence we can solve for s in terms of t(inverse function theorem). Hence we can use s as a new parameter to represent the curve C.

Example 12.3.9. For the helix $\mathbf{r}(t) = (a\cos t, a\sin t, bt)$, we can find a new parametrization by s as follows:

$$s(t) = \int_0^t \|\mathbf{r}'(\tau)\|\, d\tau = \int_0^t \sqrt{a^2+b^2}d\tau = \sqrt{a^2+b^2}\ t,$$

so that

$$s = \sqrt{a^2+b^2}\ t, \text{ or } t = \frac{s}{\sqrt{a^2+b^2}}.$$

Hence

$$\mathbf{r}(t(s)) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}}\right).$$

Definition 12.3.10. The **unit tangent vector T** of the path $\mathbf{r}$ is the normalized velocity vector

$$\mathbf{T} = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Example 12.3.11. For the helix $\mathbf{r} = (a\cos t, a\sin t, bt)$, we have

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}}{\sqrt{a^2+b^2}}.$$

Example 12.3.12. For the curve $\mathbf{r} = (t, t^2, t^3)$, we have

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}.$$

$$\mathbf{T} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}}{\sqrt{1+4t^2+9t^4}}.$$

But arclength is not easy to compute:

$$s(t) = \int_0^t \sqrt{1+4t^2+9t^4}\, dt.$$

Example 12.3.13 (Change of the position $\mathbf{r}$ vector w.r.t arclength)**.** In general, finding a parametrization by arclength parameter s is not a simple task. However, it has important meaning: Assume $\mathbf{r}(s)$ be a parametrization by arclength parameter. Then by the chain rule and property of arclength parameter, we have

$$\begin{aligned}\mathbf{r}'(t) &= \mathbf{r}'(s)\frac{ds}{dt}\\ &= \mathbf{r}'(s)\|\mathbf{r}'(t)\|.\end{aligned}$$

Since $\|\mathbf{r}'(t)\| \neq 0$, we have

$$\mathbf{r}'(s) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \left(\text{i.e., } \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} = \mathbf{T} \right).$$

Thus $\mathbf{r}(s)$ has always unit speed (i.e., $\mathbf{r}'(s)$ always has a unit length). The two parametrization $(a\cos t, a\sin t)$ and $(a\cos 2\pi t, a\sin 2\pi t)$ have different speeds along the same circle. For the first one, $\mathbf{r}'(t) = (-a\sin t, a\cos t)$. So

$$s(t) = \int_0^t \sqrt{a^2}d\tau = at.$$

So

$$(a\cos t, a\sin t) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}).$$

While for the second one, $\mathbf{r}'(t) = (-2a\pi\sin t, 2a\pi\cos t)$. So

$$s(t) = \int_0^t 2a\pi d\tau = 2a\pi t.$$

Solving $t = s/2a\pi$. So

$$(a\cos 2\pi t, a\sin 2\pi t) = (a\cos\frac{s}{a}, a\sin\frac{s}{a}).$$

So the parametrization by the arc length parameter is the same. In fact, it is independent of any parametrization(Why?)

12.4 Curvature and Normal vectors of a Curve

To measure how the curve bends we need to define the following:

Definition 12.4.1. The **curvature** of a path $\mathbf{r}$ is the rate of change of unit tangent vector $\mathbf{T}$ per unit of length along the path. In other words,

$$\kappa(t) = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{\|d\mathbf{T}/dt\|}{ds/dt} = \frac{1}{\|\mathbf{v}\|}\left\|\frac{d\mathbf{T}}{dt}\right\|.$$

If $\|\frac{d\mathbf{T}}{ds}\|$ is large at some point P, the curve turns sharply, and the curvature is large there.

Example 12.4.2. Consider a line $\mathbf{r}(t) = \mathbf{a} + t\mathbf{v}$ for some constant vector $\mathbf{a}$. $\mathbf{r}'(t) = \mathbf{v}$, and $\mathbf{T} = \mathbf{v}/\|\mathbf{v}\|$ is a constant vector. So

$$\kappa = 0.$$

Lemma 12.4.3.

$$\frac{d}{dt}|\mathbf{v}| = \frac{\mathbf{v}}{|\mathbf{v}|}\mathbf{v}' = sgn(\mathbf{v})\mathbf{v}'.$$

Proof.

$$\frac{d}{dt}|\mathbf{v}|^2 = 2|\mathbf{v}|\frac{d}{dt}|\mathbf{v}| = \frac{d}{dt}(\mathbf{v}\cdot\mathbf{v}) = 2\mathbf{v}\cdot\mathbf{v}'.$$

□

Circular Orbits

Consider a particle of mass m moving at constant speed s in a circular path of radius r_0. We can represent its motion (in the plane) as

$$\mathbf{r}(t) = (r_0\cos t, r_0\sin t).$$

Since speed is $\|\mathbf{r}'(t)\| = v = r_0$. So the motion is described as

$$\mathbf{v} = \mathbf{r}'(t) = (-r_0\sin t, r_0\cos t), \ \|\mathbf{v}\| = r_0.$$

$$\begin{aligned}
\mathbf{T} &= \frac{\mathbf{v}}{\|\mathbf{v}\|} = (-\sin t, \cos t) \\
\frac{d\mathbf{T}}{dt} &= (-\cos t, -\sin t) \\
\|\frac{d\mathbf{T}}{dt}\| &= 1.
\end{aligned}$$

Hence

$$\kappa = \frac{1}{\|\mathbf{v}\|}\left\|\frac{d\mathbf{T}}{dt}\right\| = \frac{1}{r_0} = \frac{1}{radius}.$$

Since $\mathbf{T}(t)$ is a vector whose length is constant, we have $1 = \|\mathbf{T}(t)\|^2 = \mathbf{T}(t)\cdot\mathbf{T}(t)$. Taking the derivative of constant is zero. Hence

$$0 = \frac{d}{dt}[\mathbf{T}(t)\cdot\mathbf{T}(t)] = \mathbf{T}'(t)\cdot\mathbf{T}(t) + \mathbf{T}(t)\cdot\mathbf{T}'(t) = 2\mathbf{T}(t)\cdot\mathbf{T}'(t).$$

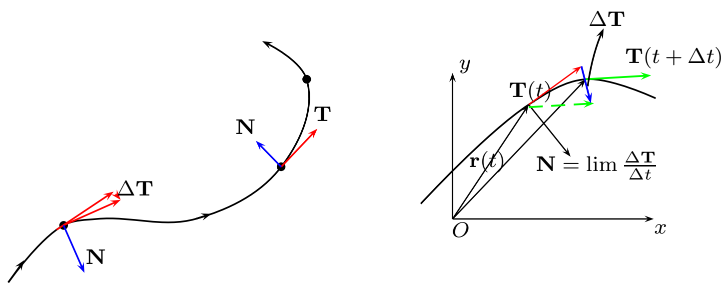


Figure 12.11: **T** turns in the direction of **N**

Thus $\mathbf{T}'(t)$ is perpendicular to $\mathbf{T}(t)$ for all t.

The vector $\frac{d\mathbf{T}}{dt}$ turns in the direction along which the curve turns.

Definition 12.4.4. At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}.$$

The second equality is verified as follows.

$$\begin{aligned} \mathbf{N} &= \frac{d\mathbf{T}/ds}{\|d\mathbf{T}/ds\|} \text{ (use Chain rule)} \\ &= \frac{(d\mathbf{T}/dt)(dt/ds)}{\|d\mathbf{T}/dt\|(dt/ds)} \ (\because dt/ds > 0) \\ &= \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}. \end{aligned}$$

Circle of Curvature for Plane curves

The **circle of curvature** or **osculating circle** at a point P is defined when $\kappa \neq 0$. It is a circle that

(1) has the same tangent line as the curve has

(2) has the same curvature as the curve has

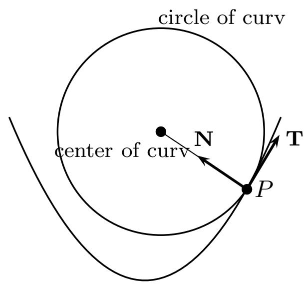


Figure 12.12: Circle of Curvature

(3) has center in the concave side

The **radius of curvature** of the curve at P is the radius of the circle of curvature. (i.e, $1/\kappa$) Similarly, the **osculating plane** can be defined.

Example 12.4.5. Find the osculating circle of parabola $y = x^2$ at the origin.

sol. We parameterize the parabola by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}.$$

Find the osculating circle of parabola $y = x^2$ at the origin.

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \\
|\mathbf{v}| &= \sqrt{1 + 4t^2} \\
\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = (1 + 4t^2)^{-1/2}\mathbf{i} + 2t(1 + 4t^2)^{-1/2}\mathbf{j}.
\end{aligned}$$

Hence

$$\frac{d\mathbf{T}}{dt} = -4t(1 + 4t^2)^{-3/2}\mathbf{i} + [2(1 + 4t^2)^{-1/2} - 8t^2(1 + 4t^2)^{-3/2}]\mathbf{j}.$$

When $t = 0$,

$$\begin{aligned}
\kappa &= \frac{1}{|\mathbf{v}(0)|}\left|\frac{d\mathbf{T}}{dt}(0)\right| \\
&= \sqrt{0^2 + 2^2} = 2
\end{aligned}$$

Now the normal $\mathbf{N} = \mathbf{j}$. Hence the center is at $(0, 1/2)$ and the circle is

$$(x-0)^2 + (y-\frac{1}{2})^2 = (\frac{1}{2})^2.$$

□

Curvature and normal vectors for Space curves

The **curvature** and the **principal unit normal** vector for a smooth curve of a space curve given by $\mathbf{r}$ defined to be the same as plane curve.

$$\kappa = \left\|\frac{d\mathbf{T}}{ds}\right\| = \frac{1}{\|\mathbf{v}\|}\left\|\frac{d\mathbf{T}}{dt}\right\| \qquad (12.5)$$

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds} = \frac{d\mathbf{T}/dt}{\|d\mathbf{T}/dt\|}. \qquad (12.6)$$

Example 12.4.6. Find the curvature for the helix

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + bt\mathbf{k}, a, b > 0.$$

sol.

$$\begin{aligned}
\mathbf{v} &= -(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k} \\
|\mathbf{v}| &= \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2} = \sqrt{a^2+b^2} \\
\mathbf{T} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{a^2+b^2}}[-(a\sin t)\mathbf{i} + (a\cos t)\mathbf{j} + b\mathbf{k}].
\end{aligned}$$

Hence

$$\begin{aligned}
\kappa &= \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right| \\
&= \frac{1}{\sqrt{a^2+b^2}}\left|\frac{1}{\sqrt{a^2+b^2}}[-(a\cos t)\mathbf{i} - (a\sin t)\mathbf{j}]\right| \\
&= \frac{a}{a^2+b^2}\,|[-\cos t\mathbf{i} - \sin t\mathbf{j}]| \\
&= \frac{a}{a^2+b^2}.
\end{aligned}$$

□

Example 12.4.7. Find the normal $\mathbf{N}$ for the helix above.

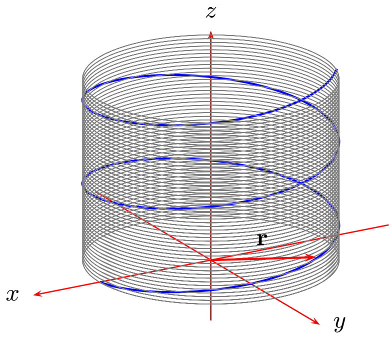


Figure 12.13: Helix

$$
\begin{aligned}
\frac{d\mathbf{T}}{dt} &= -\frac{1}{\sqrt{a^2+b^2}}[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}] \\
|\frac{d\mathbf{T}}{dt}| &= \frac{1}{\sqrt{a^2+b^2}}\sqrt{a^2\cos^2 t + b^2\sin^2 t} = \frac{a}{\sqrt{a^2+b^2}} \\
\mathbf{N} &= \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} = -\frac{\sqrt{a^2+b^2}}{a}\frac{1}{\sqrt{a^2+b^2}}[(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}] \\
&= -(\cos t)\mathbf{i} - (\sin t)\mathbf{j}.
\end{aligned}
$$

Hence **N** is always lying in the xy - plane and pointing toward z axis.

12.5 Tangent and Normal components of acceleration

Given a curve, we have seen the unit tangent vector **T** and the unit normal vector **N**. Using these we can define a third vector **B** (called **binormal**, normal to the plane of **T** and **N**) by

$$\mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

The three vectors **T**, **N** and **B** form an orthogonal coordinate system (called **TNB frame** or Frenet (1816-1900) frame) and is useful in studying an object moving on the curve.

We see

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \mathbf{T}\frac{ds}{dt} \\
\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\mathbf{T}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{dt} \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\frac{d\mathbf{T}}{ds}\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\left(\kappa\mathbf{N}\frac{ds}{dt}\right) \\
&= \frac{d^2s}{dt^2}\mathbf{T} + \kappa\left(\frac{ds}{dt}\right)^2\mathbf{N}.
\end{aligned}$$

Definition 12.5.1. If acceleration vector is written as

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} \tag{12.7}$$

then

$$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}|v| \quad \text{and} \quad a_N = \kappa\left(\frac{ds}{dt}\right)^2 = \kappa|\mathbf{v}|^2 \tag{12.8}$$

are **tangential** and **normal** components of acceleration. We see

$$\boxed{a_N = \sqrt{\|\mathbf{a}\|^2 - a_T^2}} \tag{12.9}$$

Example 12.5.2. Without finding **T** and **N**, write the acceleration of the motion (involute, see Fig 12.27 of the book)

$$\mathbf{r}(t) = (\cos t + t\sin t)\mathbf{i} + (\sin t - t\cos t)\mathbf{j}, \ t > 0$$

in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$.

sol. We can use the formula (12.8) (or (12.9) also).

$$\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = (-\sin t + \sin t + t\cos t)\mathbf{i} + (\cos t - \cos t + t\sin t)\mathbf{j} \\
&= (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} \\
|\mathbf{v}| &= \sqrt{t^2\cos^2 t + t^2\sin^2 t} = t \\
a_T &= \frac{d}{dt}\|\mathbf{v}\| = \frac{d}{dt}(t) = 1.
\end{aligned}$$

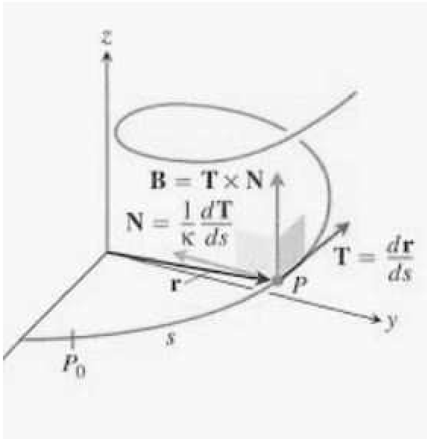


Figure 12.14: Binormal

$$\begin{aligned} \mathbf{a} &= (\cos t - t\sin t)\mathbf{i} + (\sin t + t\cos t)\mathbf{j} \\ |\mathbf{a}|^2 &= t^2 + 1 \\ a_N &= \sqrt{\|\mathbf{a}\|^2 - a_T^2} = \sqrt{t^2+1-1} = t. \end{aligned}$$

Thus

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N} = \mathbf{T} + t\mathbf{N}.$$

□

Torsion

How does $d\mathbf{B}/ds$ behaves in relation to $\mathbf{T}, \mathbf{N}, \mathbf{B}$?

$$\frac{d\mathbf{B}}{ds} = \frac{d(\mathbf{T}\times\mathbf{N})}{ds} = \frac{d\mathbf{T}}{ds}\times\mathbf{N} + \mathbf{T}\times\frac{d\mathbf{N}}{ds} = 0 + \mathbf{T}\times\frac{d\mathbf{N}}{ds}.$$

Hence we can see $d\mathbf{B}/ds$ is orthogonal to $\mathbf{T}$. We can also see $d\mathbf{B}/ds$ is orthogonal to $\mathbf{B}$. Since $\mathbf{B}\cdot\mathbf{B} = 1$,

$$0 = \frac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 2\mathbf{B}\cdot\frac{d\mathbf{B}}{ds}.$$

Thus (since it is orthogonal to both $\mathbf{T}$ and $\mathbf{B}$) it is a scalar multiple of $\mathbf{N}$.

Hence we have

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

for some scalar τ. This τ is called **torsion** and one can easily see that it satisfies

$$\boxed{\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N}.}$$

In summary,

(1) $\kappa = |d\mathbf{T}/ds|$ is the rate at which the normal plane turns about the point P as the point moves along the curve.

(2) $\tau = -(d\mathbf{B})/ds)\mathbf{N}$ is the rate at which the osculating plane turns about $\mathbf{T}$ as the point moves along the curve.

Formula for computing the curvature and torsion

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= \left(\frac{ds}{dt}\mathbf{T}\right) \times \left[\frac{d^2 s}{dt^2}\mathbf{T} + \kappa \left(\frac{ds}{dt}\right)^2 \mathbf{N}\right] \\
&= \left(\frac{ds}{dt}\frac{d^2 s}{dt^2}\right)(\mathbf{T} \times \mathbf{T}) + \kappa \left(\frac{ds}{dt}\right)^3 (\mathbf{T} \times \mathbf{N}) \\
&= \kappa \left(\frac{ds}{dt}\right)^3 \cdot \mathbf{B}.
\end{aligned}$$

Hence

$$|\mathbf{v} \times \mathbf{a}| = \kappa \left|\frac{ds}{dt}\right|^3 |\mathbf{B}| = \kappa |\mathbf{v}|^3.$$

$$\boxed{\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}} \tag{12.10}$$

Skip the formula for torsion (see the book)

12.6 Velocity and Acceleration in Polar Coordinates

Motion in Polar and Cylindrical coordinates

When a particle moves along a curve given by polar coordinates, we express, position, velocity and acceleration in terms of moving unit vectors (as shown in Figure)

$$\mathbf{u}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}, \quad \mathbf{u}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}.$$

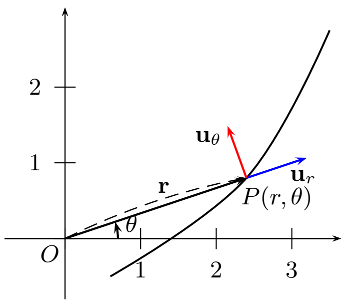


Figure 12.15: A curve in polar coord

The vector $\mathbf{u_r}$ points along the position vector $\vec{OP} = \mathbf{r} = r\mathbf{u_r}$, the vector $\mathbf{u}_\theta$, orthogonal to $\mathbf{u}_r$, points in the direction of increasing θ.

We see

$$\begin{aligned}\frac{d\mathbf{u}_r}{d\theta} &= -(\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j} = \mathbf{u}_\theta \\ \frac{d\mathbf{u}_\theta}{d\theta} &= -(\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} = -\mathbf{u}_r\end{aligned}$$

The derivative w.r.t t is denoted by $\dot{\mathbf{u}}$. By chain rule

$$\dot{\mathbf{u}}_r = \frac{d\mathbf{u}_r}{d\theta}\dot{\theta} = \dot{\theta}\mathbf{u}_\theta, \quad \dot{\mathbf{u}}_\theta = \frac{d\mathbf{u}_\theta}{d\theta}\dot{\theta} = -\dot{\theta}\mathbf{u_r}. \tag{12.11}$$

Hence we can express the velocity in terms of $\mathbf{u}_r$ and $\mathbf{u}_\theta$ as

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d}{dt}(r\mathbf{u_r}) = \dot{r}\mathbf{u_r} + r\dot{\mathbf{u}}_\mathbf{r} = \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta.$$

The acceleration is

$$\mathbf{a} = \dot{\mathbf{v}} = (\ddot{r}\mathbf{u_r} + \dot{r}\dot{\mathbf{u}}_\mathbf{r}) + (\dot{r}\dot{\theta}\mathbf{u}_\theta + r\ddot{\theta}\mathbf{u}_\theta + r\dot{\theta}\dot{\mathbf{u}}_\theta) \tag{12.12}$$

If we use (12.11) we have

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta. \tag{12.13}$$

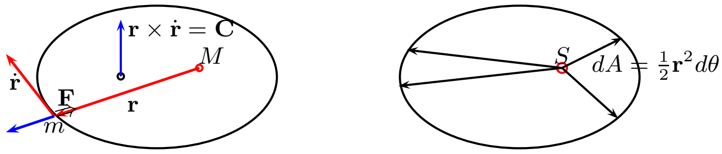


Figure 12.16: Gravity: Planet sweeps equal area in equal times

$$\boxed{\begin{array}{rcl} \mathbf{r} & = & r\mathbf{u_r} \\ \mathbf{v} & = & \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta \\ \mathbf{a} & = & (\ddot{r} - r\dot{\theta}^2)\mathbf{u_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{u}_\theta \end{array}} \tag{12.14}$$

Planets Move in Planes

Newton's Law of gravitation: If $\mathbf{r}$ is the radius vector from the center of a sun of mass M to the center of the planet of mass m, then the force acting the two objects is

$$\mathbf{F} = -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

Combining with te Newton's second law, $\mathbf{F} = m\ddot{\mathbf{r}}$, we get

$$m\ddot{\mathbf{r}} = -\frac{GmM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

$$\ddot{\mathbf{r}} = -\frac{GM}{|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

The planet is always accelerated toward the center of the sun. And $\mathbf{r} \times \ddot{\mathbf{r}} = 0$.

Hence

$$\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0$$

it follows that

$$\mathbf{r} \times \dot{\mathbf{r}} = \mathbf{C}. \tag{12.15}$$

This tells us that $\mathbf{r}$ and $\dot{\mathbf{r}}$ always lie in the plane perp. to the vector $\mathbf{C}$.

Kepler's second law, Fig 12.16

Assume the plane of the planet is xy-plane so that $\mathbf{C}$ is in the direction of $\mathbf{k}$. Also, we introduce a polar coordinate so that along the line $\theta = 0$, the direction $\mathbf{r}$ when $t = 0$ becomes minimum so $r(0) = r_0$ and (using $\mathbf{v} = \dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta$)

$$\dot{r}|_{t=0} = 0 \; and \; v_0 = |\mathbf{v}|_{t=0} = [r\dot{\theta}]_{t=0}.$$

We use (12.14). Then

$$\begin{aligned} \mathbf{C} &= \mathbf{r} \times \dot{\mathbf{r}} = \mathbf{r} \times \mathbf{v} \\ &= r\mathbf{u_r} \times (\dot{r}\mathbf{u_r} + r\dot{\theta}\mathbf{u}_\theta) \\ &= 0 + r^2\dot{\theta}(\mathbf{u_r} \times \mathbf{u}_\theta) \\ &= r^2\dot{\theta}\mathbf{k}. \end{aligned}$$

Set $t = 0$.

$$\mathbf{C} = r_0 v_0 \mathbf{k}.$$

Subst this into above eq. we get

$$r^2\dot{\theta}\mathbf{k} = r_0 v_0 \mathbf{k}, \; or \; r^2\dot{\theta} = r_0 v_0.$$

Since the area of the sector by polar coordinate (section 11.5) is

$$dA = \frac{1}{2} r^2 d\theta.$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} r_0 v_0 = const$$

which means the change of area is constant(Kepler's second law).